

List of Symbols, Notations and Data

$B(n, p)$: Binomial distribution with n trials and success probability p ; $n \in \{1, 2, \dots\}$ and $p \in (0, 1)$

$U(a, b)$: Uniform distribution on the interval (a, b) , $-\infty < a < b < \infty$

$N(\mu, \sigma^2)$: Normal distribution with mean μ and variance σ^2 , $\mu \in (-\infty, \infty)$, $\sigma > 0$

$P(A)$: Probability of the event A

Poisson(λ): Poisson distribution with mean λ , $\lambda > 0$

$E(X)$: Expected value (mean) of the random variable X

If $Z \sim N(0, 1)$, then $P(Z \leq 1.96) = 0.975$ and $P(Z \leq 0.54) = 0.7054$

\mathbb{Z} : Set of integers

\mathbb{Q} : Set of rational numbers

\mathbb{R} : Set of real numbers

\mathbb{C} : Set of complex numbers

\mathbb{Z}_n : The cyclic group of order n

$\mathbb{F}[x]$: Polynomial ring over the field \mathbb{F}

$C[0, 1]$: Set of all real valued continuous functions on the interval $[0, 1]$

$C^1[0, 1]$: Set of all real valued continuously differentiable functions on the interval $[0, 1]$

ℓ_2 : Normed space of all square-summable real sequences

$L^2[0, 1]$: Space of all square-Lebesgue integrable real valued functions on the interval $[0, 1]$

$(C[0, 1], \| \cdot \|_2)$: The space $C[0, 1]$ with $\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}}$

$(C[0, 1], \| \cdot \|_\infty)$: The space $C[0, 1]$ with $\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$

V^\perp : The orthogonal complement of V in an inner product space

\mathbb{R}^n : n -dimensional Euclidean space

Usual metric d on \mathbb{R}^n is given by $d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = (\sum_{i=1}^n (x_i - y_i)^2)^{1/2}$

I_n : The $n \times n$ identity matrix (I : the identity matrix when order is NOT specified)

$o(g)$: The order of the element g of a group

Q. 1 – Q. 25 carry one mark each.

Q.1 Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be a linear map defined by

$$T(x, y, z, w) = (x + z, 2x + y + 3z, 2y + 2z, w).$$

Then the rank of T is equal to _____

Q.2 Let M be a 3×3 matrix and suppose that 1, 2 and 3 are the eigenvalues of M . If

$$M^{-1} = \frac{M^2}{\alpha} - M + \frac{11}{\alpha} I_3$$

for some scalar $\alpha \neq 0$, then α is equal to _____

Q.3 Let M be a 3×3 singular matrix and suppose that 2 and 3 are eigenvalues of M . Then the number of linearly independent eigenvectors of $M^3 + 2M + I_3$ is equal to _____

Q.4 Let M be a 3×3 matrix such that $M \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}$ and suppose that $M^3 \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Then $|\alpha|$ is equal to _____

Q.5 Let $f: [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$f(x) = \int_0^x \sin^2(t^2) dt.$$

Then the function f is

- (A) uniformly continuous on $[0, 1)$ but NOT on $(0, \infty)$
- (B) uniformly continuous on $(0, \infty)$ but NOT on $[0, 1)$
- (C) uniformly continuous on both $[0, 1)$ and $(0, \infty)$
- (D) neither uniformly continuous on $[0, 1)$ nor uniformly continuous on $(0, \infty)$

Q.6

Consider the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \begin{cases} \frac{1}{3^n} & \text{if } n \text{ is even} \\ \frac{1}{5^n} & \text{if } n \text{ is odd.} \end{cases}$

The radius of convergence of the series is equal to _____

Q.7

Let $C = \{z \in \mathbb{C} : |z - i| = 2\}$. Then $\frac{1}{2\pi} \oint_C \frac{z^2 - 4}{z^2 + 4} dz$ is equal to _____

Q.8

Let $X \sim B(5, \frac{1}{2})$ and $Y \sim U(0, 1)$. Then $\frac{P(X+Y \leq 2)}{P(X+Y \geq 5)}$ is equal to _____

Q.9 Let the random variable X have the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{2} & \text{if } 0 \leq x < 1 \\ \frac{3}{5} & \text{if } 1 \leq x < 2 \\ \frac{1}{2} + \frac{x}{8} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } x \geq 3. \end{cases}$$

Then $P(2 \leq X < 4)$ is equal to _____

Q.10 Let X be a random variable having the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4} & \text{if } 0 \leq x < 1 \\ \frac{1}{3} & \text{if } 1 \leq x < 2 \\ \frac{1}{2} & \text{if } 2 \leq x < \frac{11}{3} \\ 1 & \text{if } x \geq \frac{11}{3}. \end{cases}$$

Then $E(X)$ is equal to _____

Q.11 In an experiment, a fair die is rolled until two sixes are obtained in succession. The probability that the experiment will end in the fifth trial is equal to

- (A) $\frac{125}{6^5}$ (B) $\frac{150}{6^5}$ (C) $\frac{175}{6^5}$ (D) $\frac{200}{6^5}$

Q.12 Let $x_1 = 2.2$, $x_2 = 4.3$, $x_3 = 3.1$, $x_4 = 4.5$, $x_5 = 1.1$ and $x_6 = 5.7$ be the observed values of a random sample of size 6 from a $U(\theta - 1, \theta + 4)$ distribution, where $\theta \in (0, \infty)$ is unknown. Then a maximum likelihood estimate of θ is equal to

- (A) 1.8 (B) 2.3 (C) 3.1 (D) 3.6

Q.13 Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ be the open unit disc in \mathbb{R}^2 with boundary $\partial\Omega$. If $u(x, y)$ is the solution of the Dirichlet problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{in } \Omega \\ u(x, y) &= 1 - 2y^2 && \text{on } \partial\Omega, \end{aligned}$$

then $u\left(\frac{1}{2}, 0\right)$ is equal to

- (A) -1 (B) $\frac{-1}{4}$ (C) $\frac{1}{4}$ (D) 1

- Q.14 Let $c \in \mathbb{Z}_3$ be such that $\frac{\mathbb{Z}_3[X]}{\langle X^3 + cX + 1 \rangle}$ is a field. Then c is equal to _____
- Q.15 Let $V = C^1[0, 1]$, $X = (C[0, 1], \|\cdot\|_\infty)$ and $Y = (C[0, 1], \|\cdot\|_2)$. Then V is
 (A) dense in X but NOT in Y
 (B) dense in Y but NOT in X
 (C) dense in both X and Y
 (D) neither dense in X nor dense in Y
- Q.16 Let $T : (C[0, 1], \|\cdot\|_\infty) \rightarrow \mathbb{R}$ be defined by $T(f) = \int_0^1 2xf(x) dx$ for all $f \in C[0, 1]$. Then $\|T\|$ is equal to _____
- Q.17 Let τ_1 be the usual topology on \mathbb{R} . Let τ_2 be the topology on \mathbb{R} generated by $B = \{[a, b) \subset \mathbb{R} : -\infty < a < b < \infty\}$. Then the set $\{x \in \mathbb{R} : 4 \sin^2 x \leq 1\} \cup \left\{\frac{\pi}{2}\right\}$ is
 (A) closed in (\mathbb{R}, τ_1) but NOT in (\mathbb{R}, τ_2)
 (B) closed in (\mathbb{R}, τ_2) but NOT in (\mathbb{R}, τ_1)
 (C) closed in both (\mathbb{R}, τ_1) and (\mathbb{R}, τ_2)
 (D) neither closed in (\mathbb{R}, τ_1) nor closed in (\mathbb{R}, τ_2)
- Q.18 Let X be a connected topological space such that there exists a non-constant continuous function $f : X \rightarrow \mathbb{R}$, where \mathbb{R} is equipped with the usual topology. Let $f(X) = \{f(x) : x \in X\}$. Then
 (A) X is countable but $f(X)$ is uncountable
 (B) $f(X)$ is countable but X is uncountable
 (C) both $f(X)$ and X are countable
 (D) both $f(X)$ and X are uncountable
- Q.19 Let d_1 and d_2 denote the usual metric and the discrete metric on \mathbb{R} , respectively. Let $f : (\mathbb{R}, d_1) \rightarrow (\mathbb{R}, d_2)$ be defined by $f(x) = x$, $x \in \mathbb{R}$. Then
 (A) f is continuous but f^{-1} is NOT continuous
 (B) f^{-1} is continuous but f is NOT continuous
 (C) both f and f^{-1} are continuous
 (D) neither f nor f^{-1} is continuous
- Q.20 If the trapezoidal rule with single interval $[0, 1]$ is exact for approximating the integral $\int_0^1 (x^3 - cx^2) dx$, then the value of c is equal to _____
- Q.21 Suppose that the Newton-Raphson method is applied to the equation $2x^2 + 1 - e^{x^2} = 0$ with an initial approximation x_0 sufficiently close to zero. Then, for the root $x = 0$, the order of convergence of the method is equal to _____

Q.22 The minimum possible order of a homogeneous linear ordinary differential equation with real constant coefficients having $x^2 \sin(x)$ as a solution is equal to _____

Q.23 The Lagrangian of a system in terms of polar coordinates (r, θ) is given by

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - m g r (1 - \cos(\theta)),$$

where m is the mass, g is the acceleration due to gravity and \dot{s} denotes the derivative of s with respect to time. Then the equations of motion are

(A) $2 \ddot{r} = r \dot{\theta}^2 - g (1 - \cos(\theta)), \quad \frac{d}{dt}(r^2 \dot{\theta}) = -g r \sin(\theta)$

(B) $2 \ddot{r} = r \dot{\theta}^2 + g (1 - \cos(\theta)), \quad \frac{d}{dt}(r^2 \dot{\theta}) = -g r \sin(\theta)$

(C) $2 \ddot{r} = r \dot{\theta}^2 - g (1 - \cos(\theta)), \quad \frac{d}{dt}(r^2 \dot{\theta}) = g r \sin(\theta)$

(D) $2 \ddot{r} = r \dot{\theta}^2 + g (1 - \cos(\theta)), \quad \frac{d}{dt}(r^2 \dot{\theta}) = g r \sin(\theta)$

Q.24 If $y(x)$ satisfies the initial value problem

$$(x^2 + y)dx = x dy, \quad y(1) = 2,$$

then $y(2)$ is equal to _____

Q.25 It is known that Bessel functions $J_n(x)$, for $n \geq 0$, satisfy the identity

$$e^{\frac{x}{2}(t - \frac{1}{t})} = J_0(x) + \sum_{n=1}^{\infty} J_n(x) \left(t^n + \frac{(-1)^n}{t^n} \right)$$

for all $t > 0$ and $x \in \mathbb{R}$. The value of $J_0\left(\frac{\pi}{3}\right) + 2 \sum_{n=1}^{\infty} J_{2n}\left(\frac{\pi}{3}\right)$ is equal to _____

Q. 26 – Q. 55 carry two marks each.

Q.26 Let X and Y be two random variables having the joint probability density function

$$f(x, y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the conditional probability $P\left(X \leq \frac{2}{3} \mid Y = \frac{3}{4}\right)$ is equal to

(A) $\frac{5}{9}$

(B) $\frac{2}{3}$

(C) $\frac{7}{9}$

(D) $\frac{8}{9}$

Q.27 Let $\Omega = (0, 1]$ be the sample space and let $P(\cdot)$ be a probability function defined by

$$P((0, x]) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < \frac{1}{2} \\ x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then $P\left(\left\{\frac{1}{2}\right\}\right)$ is equal to _____

- Q.28 Let X_1, X_2 and X_3 be independent and identically distributed random variables with $E(X_1) = 0$ and $E(X_1^2) = \frac{15}{4}$. If $\psi : (0, \infty) \rightarrow (0, \infty)$ is defined through the conditional expectation

$$\psi(t) = E(X_1^2 \mid X_1^2 + X_2^2 + X_3^2 = t), \quad t > 0,$$

then $E(\psi((X_1 + X_2)^2))$ is equal to _____

- Q.29 Let $X \sim \text{Poisson}(\lambda)$, where $\lambda > 0$ is unknown. If $\delta(X)$ is the unbiased estimator of $g(\lambda) = e^{-\lambda}(3\lambda^2 + 2\lambda + 1)$, then $\sum_{k=0}^{\infty} \delta(k)$ is equal to _____

- Q.30 Let X_1, \dots, X_n be a random sample from $N(\mu, 1)$ distribution, where $\mu \in \{0, \frac{1}{2}\}$. For testing the null hypothesis $H_0: \mu = 0$ against the alternative hypothesis $H_1: \mu = \frac{1}{2}$, consider the critical region

$$R = \left\{ (x_1, x_2, \dots, x_n) : \sum_{i=1}^n x_i > c \right\},$$

where c is some real constant. If the critical region R has size 0.025 and power 0.7054, then the value of the sample size n is equal to _____

- Q.31 Let X and Y be independently distributed central chi-squared random variables with degrees of freedom $m (\geq 3)$ and $n (\geq 3)$, respectively. If $E\left(\frac{X}{Y}\right) = 3$ and $m + n = 14$, then $E\left(\frac{Y}{X}\right)$ is equal to

- (A) $\frac{2}{7}$ (B) $\frac{3}{7}$ (C) $\frac{4}{7}$ (D) $\frac{5}{7}$

- Q.32 Let X_1, X_2, \dots be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = \frac{1}{4}$ and $P(X_1 = 2) = \frac{3}{4}$. If $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$, for $n = 1, 2, \dots$, then $\lim_{n \rightarrow \infty} P(\bar{X}_n \leq 1.8)$ is equal to _____

- Q.33 Let $u(x, y) = 2f(y) \cos(x - 2y)$, $(x, y) \in \mathbb{R}^2$, be a solution of the initial value problem

$$\begin{aligned} 2u_x + u_y &= u \\ u(x, 0) &= \cos(x). \end{aligned}$$

Then $f(1)$ is equal to

- (A) $\frac{1}{2}$ (B) $\frac{e}{2}$ (C) e (D) $\frac{3e}{2}$

- Q.34 Let $u(x, t)$, $x \in \mathbb{R}$, $t \geq 0$, be the solution of the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x, 0) &= x \\ u_t(x, 0) &= 1. \end{aligned}$$

Then $u(2, 2)$ is equal to _____

- Q.35 Let $W = \text{Span} \left\{ \frac{1}{\sqrt{2}}(0,0,1,1), \frac{1}{\sqrt{2}}(1,-1,0,0) \right\}$ be a subspace of the Euclidean space \mathbb{R}^4 . Then the square of the distance from the point $(1,1,1,1)$ to the subspace W is equal to _____
- Q.36 Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be a linear map such that the null space of T is $\{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0\}$ and the rank of $(T - 4I_4)$ is 3. If the minimal polynomial of T is $x(x - 4)^\alpha$, then α is equal to _____
- Q.37 Let M be an invertible Hermitian matrix and let $x, y \in \mathbb{R}$ be such that $x^2 < 4y$. Then
 (A) both $M^2 + xM + yI$ and $M^2 - xM + yI$ are singular
 (B) $M^2 + xM + yI$ is singular but $M^2 - xM + yI$ is non-singular
 (C) $M^2 + xM + yI$ is non-singular but $M^2 - xM + yI$ is singular
 (D) both $M^2 + xM + yI$ and $M^2 - xM + yI$ are non-singular
- Q.38 Let $G = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$ with $o(x) = 4, o(y) = 2$ and $xy = yx^3$. Then the number of elements in the center of the group G is equal to
 (A) 1 (B) 2 (C) 4 (D) 8
- Q.39 The number of ring homomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_2$ to \mathbb{Z}_4 is equal to _____
- Q.40 Let $p(x) = 9x^5 + 10x^3 + 5x + 15$ and $q(x) = x^3 - x^2 - x - 2$ be two polynomials in $\mathbb{Q}[x]$. Then, over \mathbb{Q} ,
 (A) $p(x)$ and $q(x)$ are both irreducible
 (B) $p(x)$ is reducible but $q(x)$ is irreducible
 (C) $p(x)$ is irreducible but $q(x)$ is reducible
 (D) $p(x)$ and $q(x)$ are both reducible
- Q.41 Consider the linear programming problem
 Maximize $3x + 9y$,
 subject to $2y - x \leq 2$
 $3y - x \geq 0$
 $2x + 3y \leq 10$
 $x, y \geq 0$.
 Then the maximum value of the objective function is equal to _____
- Q.42 Let $S = \{ (x, \sin \frac{1}{x}) : 0 < x \leq 1 \}$ and $T = S \cup \{(0,0)\}$. Under the usual metric on \mathbb{R}^2 ,
 (A) S is closed but T is NOT closed
 (B) T is closed but S is NOT closed
 (C) both S and T are closed
 (D) neither S nor T is closed

Q.43 Let $H = \left\{ (x_n) \in \ell_2 : \sum_{n=1}^{\infty} \frac{x_n}{n} = 1 \right\}$. Then H

- (A) is bounded (B) is closed
(C) is a subspace (D) has an interior point

Q.44 Let V be a closed subspace of $L^2[0, 1]$ and let $f, g \in L^2[0, 1]$ be given by $f(x) = x$ and $g(x) = x^2$. If $V^\perp = \text{Span} \{ f \}$ and Pg is the orthogonal projection of g on V , then $(g - Pg)(x)$, $x \in [0, 1]$, is

- (A) $\frac{3}{4}x$ (B) $\frac{1}{4}x$ (C) $\frac{3}{4}x^2$ (D) $\frac{1}{4}x^2$

Q.45 Let $p(x)$ be the polynomial of degree at most 3 that passes through the points $(-2, 12)$, $(-1, 1)$, $(0, 2)$ and $(2, -8)$. Then the coefficient of x^3 in $p(x)$ is equal to _____

Q.46 If, for some $\alpha, \beta \in \mathbb{R}$, the integration formula

$$\int_0^2 p(x) dx = p(\alpha) + p(\beta)$$

holds for all polynomials $p(x)$ of degree at most 3, then the value of $3(\alpha - \beta)^2$ is equal to _____

Q.47 Let $y(t)$ be a continuous function on $[0, \infty)$ whose Laplace transform exists. If $y(t)$ satisfies

$$\int_0^t (1 - \cos(t - \tau)) y(\tau) d\tau = t^4,$$

then $y(1)$ is equal to _____

Q.48 Consider the initial value problem

$$x^2 y'' - 6y = 0, \quad y(1) = \alpha, \quad y'(1) = 6.$$

If $y(x) \rightarrow 0$ as $x \rightarrow 0^+$, then α is equal to _____

Q.49 Define $f_1, f_2: [0, 1] \rightarrow \mathbb{R}$ by

$$f_1(x) = \sum_{n=1}^{\infty} \frac{x \sin(n^2 x)}{n^2} \quad \text{and} \quad f_2(x) = \sum_{n=1}^{\infty} x^2 (1 - x^2)^{n-1}.$$

Then

- (A) f_1 is continuous but f_2 is NOT continuous
(B) f_2 is continuous but f_1 is NOT continuous
(C) both f_1 and f_2 are continuous
(D) neither f_1 nor f_2 is continuous

Q.50 Consider the unit sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and the unit normal vector $\hat{n} = (x, y, z)$ at each point (x, y, z) on S . The value of the surface integral

$$\iint_S \left\{ \left(\frac{2x}{\pi} + \sin(y^2) \right) x + \left(e^z - \frac{y}{\pi} \right) y + \left(\frac{2z}{\pi} + \sin^2 y \right) z \right\} d\sigma$$

is equal to _____

Q.51 Let $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 1000, 1 \leq y \leq 1000\}$. Define

$$f(x, y) = \frac{xy}{2} + \frac{500}{x} + \frac{500}{y}.$$

Then the minimum value of f on D is equal to _____

Q.52 Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then there exists a non-constant analytic function f on \mathbb{D} such that for all $n = 2, 3, 4, \dots$

(A) $f\left(\frac{\sqrt{-1}}{n}\right) = 0$

(B) $f\left(\frac{1}{n}\right) = 0$

(C) $f\left(1 - \frac{1}{n}\right) = 0$

(D) $f\left(\frac{1}{2} - \frac{1}{n}\right) = 0$

Q.53 Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent series expansion of $f(z) = \frac{1}{2z^2 - 13z + 15}$ in the annulus $\frac{3}{2} < |z| < 5$. Then $\frac{a_1}{a_2}$ is equal to _____

Q.54 The value of $\frac{i}{4-\pi} \int_{|z|=4} \frac{dz}{z \cos(z)}$ is equal to _____

Q.55 Suppose that among all continuously differentiable functions $y(x)$, $x \in \mathbb{R}$, with $y(0) = 0$ and $y(1) = \frac{1}{2}$, the function $y_0(x)$ minimizes the functional

$$\int_0^1 (e^{-(y'-x)} + (1+y)y') dx.$$

Then $y_0\left(\frac{1}{2}\right)$ is equal to

(A) 0

(B) $\frac{1}{8}$

(C) $\frac{1}{4}$

(D) $\frac{1}{2}$

END OF THE QUESTION PAPER